LIFTS OF SMOOTH GROUP ACTIONS TO LINE BUNDLES

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Abstract. Let X be a compact manifold with a smooth action of a compact connected Lie group G. Let $L \to X$ be a complex line bundle. Using the Cartan complex for equivariant cohomology, we give a new proof of a theorem of Hattori and Yoshida which says that the action of G lifts to L if and only if the first Chern class $c_1(L)$ of L can be lifted to an integral equivariant cohomology class in $H^2_G(X;\mathbb{Z})$, and that the different lifts of the action are classified by the lifts of $c_1(L)$ to $H^2_G(X;\mathbb{Z})$. As a corollary of our method of proof, we prove that, if the action is Hamiltonian and ∇ is a connection on L which is unitary for some metric on L and whose curvature is G-invariant, then there is a lift of the action to a certain power L^d (where d is independent of L) which leaves fixed the induced metric on L^d and the connection $\nabla^{\otimes d}$. This generalises to symplectic geometry a well known result in Geometric Invariant Theory.

1. Introduction and statement of the results

Let X be a connected smooth compact manifold with a smooth left action of a compact connected Lie group G. Our aim is to study liftings of the action of G to complex line bundles $L \to X$. Of course, it is not always possible to find such a lift. For example, if $x \in X$ has trivial stabiliser, then the restriction $L|_{Gx}$ has to be topologically trivial.

The problem which we consider is very natural and has been already studied by several people. The general question on lifting of smooth actions to principal bundles was considered for example by R. Palais and T. Stewart [PS, S]. The more concrete problem of lifting actions to complex line bundles was studied for example by B. Kostant in [Ko], where he proved that if G is simply connected, X is symplectic and the action of G is Hamiltonian, then there is always some lift of the action to L (see Theorem 4.5.1 in [Ko]), and by A. Hattori and T. Yoshida [HY] and R. Lashof [L].

Let $EG \to BG$ be the universal G-principal bundle. Fix a point $x_0 \in BG$, and denote by $\iota : X \to X_G$ the inclusion of the fibre over $x_0 \in BG$ of the Borel construction $X_G = EG \times_G X \to BG$.

Our first result is a new proof of the following theorem, which was proved in [HY]. Our method of proof is however different from theirs.

Theorem 1.1. *Let $L \to X$ be a line bundle. The action of G on X lifts to a linear action on L if and only if*

$$c_1(L) \in \iota^* H^2_G(X;\mathbb{Z}).$$

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Furthermore, if $c_1(L) \in \iota^* H^2_G(X;\mathbb{Z})$, then the different lifts of the action are classified by $\iota^{-1}(c_1(L))$.

Fix from now on a metric on L. All the connections we will take on L will be assumed to be unitary with respect to this metric, and all the actions of G on L will keep the metric fixed.

Let $\mathfrak{g} = \mathrm{Lie}(G)$. To describe lifts of the action of G and invariant connections on L we will use the Cartan model for real equivariant cohomology (see section 2 for the necessary definitions). Suppose that $c_1(L) = \iota^* l$ for some $l \in H^2_G(X;\mathbb{Z})$, and let $\alpha - \mu$ be a closed element in the Cartan complex $\Omega^*_G(X; i\mathbb{R})$ representing the class $-2\pi i l \in H^2_G(X; i\mathbb{R})$, where $\alpha \in \Omega^2(X; i\mathbb{R})^G$ and $\mu \in \Omega^0(X; i\mathfrak{g}^*)^G$. Any connection ∇ on L whose curvature is α can be combined with μ to obtain an infinitesimal lift of the action (see section 3 for definitions and Theorem 3.1), and we will study whether there is a connection ∇ which defines an infinitesimal lift that exponentiates to an action of G. Let $\mathbb{T}^\alpha$ be the gauge equivalence classes of connections on L whose curvature is α, and let $a_1 : H_1(G;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ be the map induced by the map $a_1(x) : G \ni g \mapsto gx \in X$ for any x (since X is connected, the map in homology is independent of x).

Theorem 1.2. *The set $\mathbb{T}^G_\alpha(\mu)$ of gauge equivalence classes of connections with curvature α which, combined with μ, define an infinitesimal lift which exponentiates to an action of G, is a subtorus of $\mathbb{T}^\alpha$ of dimension $b_1(X) - \dim(\mathrm{Im}\, a_1 \otimes_{\mathbb{Z}} \mathbb{R})$, where $b_1(X) = \dim H^1(X;\mathbb{R})$. In particular, if X is symplectic and the action of G is Hamiltonian, then $\mathbb{T}^G_\alpha(\mu) = \mathbb{T}^\alpha$.*

In fact, we get in this way all the possible lifts of the action. This is due to the following reasons: (1) given any lift, there is some invariant connection (just take any connection and average); (2) using the construction of differential forms in the Cartan complex representing the first equivariant Chern class $c_1^G(L)$ of L (see [BV]) we get a closed element $\alpha - \mu$ representing the class $-2\pi i c_1^G(L)$; and (3), the group G is connected, so a lift of the action is uniquely determined by the infinitesimal action on L.

The results and techniques in this paper have interesting consequences in the case in which X is symplectic and the action of G is Hamiltonian. The following corollary is an analogue of a well known result in Geometric Invariant Theory (see Corollary 1.6 in [MFK]).

Corollary 1.3. *Let G act on a symplectic manifold (X,ω) in a Hamiltonian fashion. Then there exists an integer $d \geq 1$ with the following property. For any line bundle $L \to X$ with a connection ∇ whose curvature is G invariant, there is a lift of the action of G to L^d such that the induced connection $\nabla^{\otimes d}$ on L^d is G-invariant.*

Corollary 1.4. *Let G act on a symplectic manifold (X,ω) in a Hamiltonian fashion. Let $\varepsilon > 0$. There exists a symplectic form ω' such that*

(1) ω' is preserved by G,

(2) $|\omega - \omega'|_{C^0} < \varepsilon$,

(3) the action of G on (X, ω') is Hamiltonian,

(4) there is a natural number $k > 0$, a Hermitian line bundle $L \to X$ with a unitary connection ∇ with curvature $ik\omega'$ and a linear action of G lifting the action on X and preserving ∇.

This paper is organized as follows: in section 2 we recall some facts on equivariant cohomology which we will need; in section 3 we state the relation between infinitesimal lifts and 2-forms in the Cartan complex; in section 4 we define and prove some key properties of the monodromy map M_γ, which measures how far an infinitesimal lift is to exponentiate to an action of G; in section 5 we study how to chose connections which provide liftings of the action of G; finally, in section 6 we give the proofs of Theorems 1.1 and 1.2, and Corollaries 1.3 and 1.4.

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2. Equivariant cohomology

In this section we recall some basic facts on equivariant cohomology. For more information the reader is refered to [AB, BGV, GS].

Let $\pi : X_G = EG \times_G X \to BG$ be the Borel construction of X. The equivariant cohomology of X is by definition the singular cohomology of X_G, and is denoted, for any ring R, as

$$H^*_G(X; R) := H^*(X_G; R).$$

Let $\mathfrak{g}$ be the Lie algebra of G. Let $\Omega^*(X)$ be the complex of differential forms on X. Denote by $\mathscr{X} : \mathfrak{g} \to \Gamma(TX)$ the map which assigns to any $s \in \mathfrak{g}$ the vector field on X generated by the infinitesimal action of s. So, if $f \in \Omega^0(X)$, $x \in X$ and $s \in \mathfrak{g}$, then $\mathscr{X}(s)(f)(x) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(f(e^{\varepsilon s}x) - f(x))$. Consider the complex

$$\Omega^*_G(X) = (\Omega^*(X) \otimes \mathbb{R}[\mathfrak{g}])^G$$

(as usual the supscript ^{G} means G-invariant elements) with the grading obtained from the usual grading in $\Omega^*(X)$ and twice the grading in $\mathbb{R}[\mathfrak{g}]$ given by the degree, together with the differential $d_\mathfrak{g}$ defined by

$$d_\mathfrak{g}(\eta)(s) = d(\eta(s)) + \iota_{\mathscr{X}(s)}\eta(s),$$

where $\eta \in \Omega^*_G(X)$, $s \in \mathfrak{g}$, and $\iota_v : \Omega^*(X) \to \Omega^{*-1}(X)$ is the contraction map. One can check that $d_\mathfrak{g}^2 = 0$, and the complex $(\Omega^*_G(X), d_\mathfrak{g})$ is called the Cartan complex. It will be our main tool in this paper. Note that in this paper we consider sometimes elements of $\Omega^*_G(X; i\mathbb{R}) = i\Omega^*_G(X)$.

The following classical theorem (which is a generalisation of de Rham's theorem) is proved for example as Theorem 2.5.1 in [GS].

Theorem 2.1. *There is a natural isomorphism $H^*_G(X; \mathbb{R}) \simeq H^*(\Omega^*_G(X), d_\mathfrak{g})$.*

We will need an explicit description of the isomorphism given by the theorem in degree 2, as given by the following lemma. The proof can easily be deduced from the proof of Theorem 2.5.1 in [GS].

Lemma 2.2. *Let $\alpha - \mu \in \Omega^2_G(X)$ satisfy $d_{\mathfrak{g}}(\alpha-\mu) = 0$. Let $f : \Sigma \to X_G$ be a continuous map. Let $P_f := (\pi \circ f)^* EG \to \Sigma$, and let ϕ_f be the induced section of $\pi_f : X_f = P_f \times_G X = (\pi\circ f)^* X_G \to \Sigma$. Suppose that both P_f and ϕ are smooth, and let A be a connection on P_f. We then get a projection $\pi_A : TX_f \to \operatorname{Ker} d\pi_f$, which allows to pullback α to a form $\pi_A^*\alpha \in \Omega^2(X_f)$ vanishing on the tangent vectors which are horizontal with respect to A. Then*

$$\langle [\Sigma], f^*[\alpha - \mu]\rangle = \int_\Sigma \phi_f^*(\pi_A^*\alpha) - \langle \mu, F_A\rangle,$$

where $F_A \in \Omega^2(P_f \times_{\mathrm{Ad}} \mathfrak{g})$ is the curvature of A.

Observe that a constant and equivariant map $\mu : X \to \mathfrak{g}^*$ represents an integral element $[\mu] \in H^2_G(X;\mathbb{Z})$ if and only if for any $u \in \mathfrak{g}$ such that $\exp(u) = 1$ we have $\langle \mu, u\rangle \in \mathbb{Z}$.

Finally, the map $\iota^* : H^2_G(X;\mathbb{R}) \to H^2(X;\mathbb{R})$ can be written as follows using the Cartan and de Rham complexes: if $\alpha - \mu$ is a closed element of $\Omega^2_G(X)$, then

$$\iota^*[\alpha - \mu] = [\alpha] \in H^2(X;\mathbb{R}) \tag{1}$$

(of course, since $d_{\mathfrak{g}}(\alpha - \mu) = 0$ we have $d\alpha = 0$).

3. Infinitesimal lift of the action

Let ∇ be a connection on L whose curvature $\alpha = \nabla^2 \in \Omega^2(X; i\mathbb{R})$ is G-invariant. We will call an infinitesimal lift of the action of G on X to an action on L any map $\widetilde{\mathscr{X}} : \mathfrak{g} \to \Gamma(TL)$ which satisfies[1] $\widetilde{\mathscr{X}}([s,s']) = -[\widetilde{\mathscr{X}}(s), \widetilde{\mathscr{X}}(s')]$ for any $s, s' \in \mathfrak{g}$ and such that

$$d\pi \circ \widetilde{\mathscr{X}} = \mathscr{X}.$$

Let $u = 2\pi i \in i\mathbb{R} = \mathrm{Lie}(S^1)$. Let $U_L \in \Gamma(\operatorname{Ker} d\pi)$ be the vertical tangent field generated by the infinitesimal action of s on L given by fibrewise multiplication.

Let $\mathscr{X}^\nabla : \mathfrak{g} \to \Gamma(TL)$ be the map which assigns to $s \in \mathfrak{g}$ the horizontal lift of $\mathscr{X}(s)$ obtained using ∇. The following is well known (see for example Section 3 in [Ko]).

Theorem 3.1. *Let $\mu : X \to i\mathfrak{g}^*$ be a map which satisfies $d\mu(s) = \iota_{\mathscr{X}(s)}\alpha$ for any $s \in \mathfrak{g}$. Then $\widetilde{\mathscr{X}}^{\nabla,\mu} := \mathscr{X}^\nabla + i\mu U_L$ is an infinitesimal lift of the action of G on X.*

Conversely, for any infinitesimal lift $\widetilde{\mathscr{X}}$ which leaves ∇ invariant, the function $\mu : X \to i\mathfrak{g}^*$ defined by $\widetilde{\mathscr{X}} := \mathscr{X}^\nabla + i\mu U_L$ satisfies $d\mu(s) = \iota_{\mathscr{X}(s)}\alpha$ for any $s \in \mathfrak{g}$.

Note that the condition $d\mu(s) = \iota_{\mathscr{X}(s)}\alpha$ for any $s \in \mathfrak{g}$ is equivalent to asking

$$d_{\mathfrak{g}}(\alpha - \mu) = 0.$$

[1] Since the action of G is on the left, for any $s, s' \in \mathfrak{g}$ we have $\mathscr{X}([s,s']) = -[\mathscr{X}(s), \mathscr{X}(s')]$.

4. The monodromy map M_γ

Let $(L,\nabla)\to X$ be as in the preceeding section, let $\alpha=\nabla^2$ be the curvature of ∇, and let $\mu : X\to i\mathfrak{g}^*$ be a map which satisfies $d\mu(s)=\iota_{\mathscr{X}(s)}\alpha$ for any $s\in\mathfrak{g}$. Let $\widetilde{\mathscr{X}}=\widetilde{\mathscr{X}}^{\nabla,\mu} : \mathfrak{g}\to\Gamma(TL)$ be the corresponding infinitesimal lift.

Let $\gamma : S^1\to G$ be any representation. As before, let $u=2\pi i\in i\mathbb{R}=\mathrm{Lie}(S^1)$. For any $x\in X$ and $y\in L_x$, let $\nu_x : [0,1]\to L$ be the integral line of the vector field $\widetilde{\mathscr{X}}(\gamma_*(u))$ with initial value $\nu_x(0)=y$. We then have $\nu_x(1)\in L_x$, and there is a unique $M_\gamma(x)=M_\gamma^{\nabla,\mu}(x)\in S^1$ (independent of y) such that $\nu_x(1)=M_\gamma(x)\nu_x(0)$.

The map $M_\gamma : X\to S^1$ which we have defined measures the extent to which the infinitesimal action given by (∇,μ) exponentiates to an action of the group. We will see below (viz. Lemma 5.2) that the condition $M_\gamma=1$ for any γ is enough to ensure that the infinitesimal action exponentiates.

Let $x\in X$ be any point, and let $\mathrm{Mon}^\nabla(S^1x)\in S^1$ be the monodromy of parallel transport using ∇ along the path $[0,1]\ni t\mapsto\gamma(e^{2\pi it})x$. The following formula for M can be easily proved using coordinates in a neigbourhood of S^1x (see also Theorem 2.10.1 in [Ko]):

$$M_\gamma^{\nabla,\mu}(x)=\mathrm{Mon}^\nabla(S^1x)\exp(-2\pi\langle\mu(x),\gamma_*(u)\rangle). \tag{2}$$

An easy consequence of this formula is that M is gauge invariant, i.e., for any gauge transformation $g : L\to L$,

$$M_\gamma^{g^*\nabla,\mu}(x)=M_\gamma^{\nabla,\mu}(x). \tag{3}$$

Let $\eta\in\Omega^1(X;i\mathbb{R})$, so that $\nabla+\eta$ is another connection on L. Then we also deduce from (2) that

$$M_\gamma^{\nabla+\eta,\mu}(x)=M_\gamma^{\nabla,\mu}(x)\exp\left(-\int_{S^1}\gamma_x^*\eta\right), \tag{4}$$

where $\gamma_x : S^1\to X$ maps θ to $\gamma(\theta)x\in X$.

4.1. **Cohomological interpretation of M.**

In this subsection we assume $G=S^1$ and $\gamma=\mathrm{id}$, and we denote $M=M_\gamma$. We identify $i\mathbb{R}$ with $(i\mathbb{R})^*$ by assigning to $\alpha\in i\mathbb{R}$ the map $i\mathbb{R}\ni a\mapsto\langle\alpha,a\rangle=-\alpha a/2\pi$. In particular, $\mu\in\Omega^0(X;\mathbb{R})^{S^1}$. Let us suppose that the action of S^1 on X is generically free, i.e., the isotropy group is trivial for generic $x\in X$ (if this is not true, then either the action of S^1 on X is trivial, and the results in this section are obvious, or there is a biggest common stabiliser $\{1\}\neq Z\subset S^1$, which acts freely on X, and we replace X by X/Z and S^1 by S^1/Z).

Lemma 4.1. *Let $x\in X$ be a point with trivial stabiliser, and let $O_x : S^1\to X$ be the map $O_x(\theta)=\theta x$. Assume that $O_x(S^1)$ is homologous to zero. Then, if $\frac{i}{2\pi}[\alpha-\mu]\in H^2_{S^1}(X;\mathbb{Z})$, we have $M(x)=1$.*

Proof. Let Σ_0 be a compact surface with a fixed isomorphism $\partial\Sigma_0\simeq S^1$ and compatible orientation, and let $b_0 : \Sigma_0\to X$ be a map such that $b_0|_{\partial\Sigma_0}=O_x$. Note that $\iota\circ b_0(\partial\Sigma_0)=\iota\circ O_x(S^1)$ is contained in $ES^1\times_{S^1}(S^1x)\subset X_{S^1}$. Since the action of S^1 on $S^1x\subset X$ is free, $ES^1\times_{S^1}(S^1x)$ is contractible. So, denoting the unit disk by $\mathbb{D}$,

we may take a map $B_1 : \mathbb{D} \to ES^1 \times_{S^1} (S^1 x)$ such that $c \circ B_1|_{S^1} = \iota \circ b_0|_{\partial\Sigma_0}$, where $c(\theta) = \theta^{-1}$ for $\theta \in S^1$ (such a B_1 is unique up to homotopy rel S^1). Let us patch together the maps $B_0 := \iota \circ b_0$ and B_1 to get a map $B : \Sigma := \Sigma_0 \cup_{S^1} (-\mathbb{D}) \to X_{S^1}$, where the minus sign refers to inversed orientation, so that we take the isomorphism c to identify $\partial(-\mathbb{D})$ with S^1. We claim that

$$M(x) = \exp\langle[\Sigma], B^*[-\alpha + \mu]\rangle. \tag{5}$$

Clearly, once we check the claim, the lemma is proved. To prove (5), we will apply Lemma 2.2 to the map B. Let $\pi : X_{S^1} \to BS^1$ denote the projection. Then we have $\pi \circ B_0 = \{x_0\}$, so that $P_B|_{\Sigma_0}$ (we use here the notation of Lemma 2.2) is the trivial bundle. Let us fix a trivialisation $P_B|_{\Sigma_0} \simeq \Sigma_0 \times S^1$ of it. Using the induced trivialisation $X_B|_{\Sigma_0} = \Sigma_0 \times X$, the restriction $\phi_B|_{\Sigma_0}$ is given by (id, b_0). Since $\mathbb{D}$ is contractible, P_B is obtained by patching the trivial bundles over Σ_0 and $\mathbb{D}$ through a gluing map $\rho : \partial\Sigma_0 = S^1 \to S^1$. The map $b_0|_{\partial\Sigma_0}$ has winding number 1, so the section $\phi_B|_{\Sigma_0}$ will only glue with a section of the trivial bundle $(-\mathbb{D}) \times (S^1 x) \subset (-\mathbb{D}) \times X$ if the gluing map ρ is the identity $\rho(\theta) = \theta$. Hence, the bundle P_B must have degree -1.

Take now a connection A on P_B which coincides over Σ_0 with the flat connection induced by the chosen trivialisation of $P_B|_{\Sigma_0}$. Then the curvature of A is supported in $\mathbb{D}$. By Lemma 2.2, the RHS of (5) is equal to

$$\int_\Sigma -\phi_B^*(\pi_A^*\alpha) + \langle\mu, F_A\rangle.$$

Now, by our choice of A, $\int_\Sigma \langle\mu, F_A\rangle = \int_{\mathbb{D}} \langle\mu, F_A\rangle$. On the other hand, the image of the restriction $\phi_B|_{\mathbb{D}}$ is contained in $P_B \times_{S^1} (S^1 x) \subset X_B$, and since μ is S^1 equivariant, we may write $\int_{\mathbb{D}} \langle\mu, F_A\rangle = -\frac{1}{2\pi}\mu(x) \int_{\mathbb{D}} F_A$. Finally, by Chern–Weil $\int_{\mathbb{D}} F_A = 2\pi i \deg(P_B) = 2\pi i$.

Using again that $\phi_B(\mathbb{D}) \subset P_B \times_{S^1} (S^1 x)$, we deduce that $\phi_B^*(\pi_A^*\alpha)$ vanishes on $\mathbb{D}$. Indeed, the vertical part of any tangent vector to $P_B \times_{S^1} (S^1 x)$ lies in $\mathrm{Ker}\, d\pi_B|_{P_B \times_{S^1}(S^1 x)}$, which is a real line bundle, so for any $y \in \mathbb{D}$ and $u, v \in T_y\mathbb{D}$, $\pi_A(d\phi_B(u))$ and $\pi_A(d\phi_B(v))$ are linearly dependent, and hence $\alpha(\pi_A(d\phi_B(u)), \pi_A(d\phi_B(v))) = 0$. So $\int_\Sigma -\phi_B^*(\pi_A^*\alpha) = \int_{\Sigma_0} -\phi_B^*(\pi_A^*\alpha)$. And this is equal to $\int_{\Sigma_0} -b_0^*\alpha$. Taking into account that α is the curvature of a connection ∇ on the line bundle $L \to X$, one can check that

$$\exp\left(\int_{\Sigma_0} -b_0^*\alpha\right) = \mathrm{Mon}^\nabla(S^1 x)$$

(see Theorem 1.8.1 in [Ko]). Now, using (2) and the preceeding computations we deduce

$$\begin{aligned} M(x) &= \exp\left(-\left(\int_\Sigma \phi_B^*(\pi_A^*\alpha)\right) - i\mu(x)\right) \\ &= \exp\left(\int_\Sigma -\phi_B^*(\pi_A^*\alpha) + \langle\mu(x), F_A\rangle\right) \qquad \left(\text{since} \int F_A = 2\pi i\right) \\ &= \exp\langle[\Sigma], B^*[-\alpha + \mu]\rangle. \end{aligned}$$

□

Corollary 4.2. *Assume that for some $x \in X$, $O_x(S^1)$ is homologous to zero. Then, if $\frac{i}{2\pi}[\alpha - \mu] \in H^2_{S^1}(X;\mathbb{Z})$, we have $M = 1$.*

Proof. Indeed, the condition of $O_x(S^1)$ being homologous to zero is independent of x, the function M is continuous, and by assumption the set of $x \in X$ with trivial stabiliser is dense. □

When the orbit S^1x is not homologous to zero, the map $M(x)$ will depend on the connection ∇ (and not only on α and μ), as we will see below.

Lemma 4.3. *Let $x, x' \in X$ be two points with trivial stabiliser. Then $M(x) = M(x')$.*

Proof. This can be proved either with local coordinates or using the same technique as above. We sketch the second strategy. For that, let $\rho : [0,1] \to X$ be a path such that $\rho(0) = x$ and $\rho(1) = x'$, let $\Sigma_0 = [0,1] \times S^1$ and let $b_0 : \Sigma_0 \ni (t,\theta) \mapsto \theta\rho(t)$. Glue two disks $\mathbb{D}_0$ and $\mathbb{D}_1$ to the boundary of Σ_0 with suitable orientations to get a closed oriented surface Σ, and extend the map $\iota \circ b_0$ to a map $B : \Sigma \to X_{S^1}$ just as in the preceeding lemma (i.e., so that the image of $\mathbb{D}_0$ is contained in $ES^1 \times_{S^1} (S^1x)$ and that of $\mathbb{D}_1$ in $ES^1 \times_{S^1} (S^1x')$). As before one can check that

$$M(x) - M(x') = \exp\langle [\Sigma], B^*[-\alpha + \mu]\rangle.$$

Now, however, the map B is homotopic to the trivial map, and from this the result follows. □

Corollary 4.4. *The map $M : X \to \mathbb{R}$ is constant.*

For the last lemma of this section, we return to the general situation, in which G is any compact connected Lie group.

Lemma 4.5. *Let $\gamma : S^1 \to G$ be a morphism, and let $g \in G$. We then have*

$$M_\gamma = M_{g\gamma g^{-1}}.$$

Proof. Let $\rho : S^1 \to G$ be a smooth map such that $\rho(1) = 1$ and $\rho(-1) = g$. Consider on $X \times S^1$ the action of S^1 given by

$$\theta(x,\alpha) = \rho(\alpha)\gamma(\theta)\rho(\alpha)^{-1}x \text{ for } \theta \in S^1 \text{ and } (x,\alpha) \in X \times S^1.$$

Let $\pi_1 : X \times S^1 \to X$ be the projection, and take on $X \times S^1$ the bundle π_1^*L with the connection $\nabla_{M\times S^1} = \pi_1^*\nabla$. Finally, let $\mu_{M\times S^1}(x,\alpha) = \langle \mu(x), \mathrm{Ad}(\rho(\alpha))\gamma_*(u)\rangle$. The monodromy $N = M^{\nabla_{M\times S^1}, \mu_{M\times S^1}}$ satisfies

$$N|_{X\times\{\alpha\}} = M_{\rho(\alpha)\gamma\rho(\alpha)^{-1}}.$$

Applying Corollary 4.4 to N, we deduce our result. □

5. The choice of the connection

Recall that the map $a_1 : H_1(G;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is induced from the map $a_1(x) : G \ni g \mapsto gx \in X$, where $x \in X$ is an arbitrary point. Through this section we will make the following topological assumption:

$$\operatorname{Im} a_1 \cap \operatorname{Tor} H_1(X;\mathbb{Z}) = 0. \tag{6}$$

Let $\alpha \in \Omega^2(X; i\mathbb{R})^G$ an invariant 2-form representing $-2\pi i c_1(L)$. Let $\mu \in \Omega^0(X; i\mathfrak{g}^*)^G$ satisfy $d\mu(s) = \iota_{\mathscr{X}(s)}\alpha$ for any $s \in \mathfrak{g}$, so that $\alpha - \mu \in \Omega^2_G(X; i\mathbb{R})$ is a closed form in the Cartan complex, and hence represents an equivariant cohomology class $[\alpha - \mu] \in H^2_G(X; i\mathbb{R})$.

Lemma 5.1. *Suppose that $\frac{i}{2\pi}[\alpha - \mu] \in H^2_G(X;\mathbb{Z})$. Then one can chose a connection ∇ on L whose curvature is α and such that for any morphism $\gamma : S^1 \to G$ we have $M^{\nabla,\mu}_\gamma = 1$. More preciely, the set $\mathbb{T}^G_\alpha(\mu)$ of gauge equivalence classes of connections satisfying this property is a torus of dimension $b_1(X) - \dim(\operatorname{Im} a_1 \otimes_{\mathbb{Z}} \mathbb{R})$.*

Proof. Let $\mathscr{A}_\alpha$ be the set of connections on L whose curvature is α. Let $T \subset G$ be a maximal torus. By Lemma 4.5 it is enough to consider M_γ for $\gamma : S^1 \to T$, since for any $\gamma : S^1 \to G$ there exists $g \in G$ such that $g\gamma g^{-1}(S^1) \subset T$.

Let $\mathfrak{t} = \operatorname{Lie} T$, and let $\Lambda = \operatorname{Ker}(\exp : \mathfrak{t} \to T)$, so that $T = \mathfrak{k}/\Lambda$. The morphisms $\gamma : S^1 \to T$ are in 1–1 correspondence with elements of Λ. For any $\gamma, \gamma' \in \Lambda$ we have

$$M^{\nabla,\mu}_{\gamma'+\gamma} = M^{\nabla,\mu}_{\gamma'} M^{\nabla,\mu}_{\gamma}. \tag{7}$$

To see this, observe the following. Let $y \in L$, and let $\nu(y;\cdot) : \mathfrak{t} \to L$ be the map defined as follows. For any $s \in \mathfrak{t}$, let $\nu^y_s : [0,1] \to L$ be the path such that $\nu^y_s(0) = y$ and $\nu^{y\prime}_s = \widetilde{\mathscr{X}}^{\nabla,\mu}(s)(\nu^y_s)$. Then we set $\nu(y;s) = \nu^y_s(1)$. With this definition, if $s \in \mathfrak{t}$ and $v \in T_s\mathfrak{t} \simeq \mathfrak{t}$ (use the canonical isomorphism) then $D\nu(y;s)(v) = \widetilde{\mathscr{X}}^{\nabla,\mu}(v)(s)$ (this is a consequence of $[\widetilde{\mathscr{X}}^{\nabla,\mu}(v), \widetilde{\mathscr{X}}^{\nabla,\mu}(v')] = 0$ for any $v, v' \in \mathfrak{t}$). From this it follows that

$$\nu(\nu(y;s);s') = \nu(y;s+s'), \tag{8}$$

which clearly implies (7).

Consider now the map

$$c : \Lambda \to H^1(X;\mathbb{R})$$

which sends $\gamma \in \Lambda$ to the homology class $[\gamma(S^1)]$ represented by any orbit of the S^1 action on X induced by $\gamma : S^1 \to T$. Let $\Lambda_0 = \operatorname{Ker} c$. Using condition (6), we deduce from Lemma 4.1 that for any $\gamma \in \Lambda_0$ and any connection $\nabla \in \mathscr{A}_\alpha$ we have $M^{\nabla,\mu}_\gamma = 1$. (Note that the map $\gamma^* : H^2_G(X;\mathbb{Z}) \to H^2_{S^1}(X;\mathbb{Z})$ induced by γ lifts to the Cartan complex as $\gamma^*(\alpha - \mu) = \alpha - \gamma^*(\mu)$. Let now $\Lambda_1 = \Lambda_0^\perp$. This is a free abelian module. Let $e_1, \dots, e_r \in \Lambda_1$ be a basis. By (7), if a connection $\nabla \in \mathscr{A}_\alpha$ satisfies $M^{\nabla,\mu}_{e_j} = 1$ for any $1 \le j \le r$, then $M^{\nabla,\mu}_\gamma = 1$ for all $\gamma \in \Lambda$.

Finally, by gauge invariance of $M^{\nabla,\mu}_\gamma(x)$ (3), we can consider gauge classes of connections on L rather than connections. So let $\mathbb{T}_\alpha = \mathscr{A}_\alpha / \operatorname{Map}(X, S^1)$ be the gauge

equivalence classes of connections on L with curvature α. Picking a base connection $\nabla \in \mathscr{A}_\alpha$, we can identify

$$T_\alpha = \nabla + H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}).$$

Furthermore, formula (4) implies that if $\eta \in \Omega^1(i\mathbb{R})$ satisfies $d\eta = 0$, then

$$M_{e_j}^{\nabla,\mu} = M_{e_j}^{\nabla+[\eta],\mu} - \langle [\eta], c(e_j)\rangle,$$

where $[\eta] \in H^1(X;\mathbb{R})$ is the class represented by η. On the other hand, the images by c of $e_1, \dots, e_r$ are all linearly independent. Hence, $\langle c(e_1), \dots, c(e_r)\rangle$ is a space of dimension r, so the set of gauge equivalence classes of connections $[\nabla] \in \mathbb{T}_\alpha^G(\mu)$ such that $M_\gamma^{\nabla,\mu} = 1$ for all γ is the image under the quotient

$$H^1(X;\mathbb{R}) \to H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$$

of an affine subspace of codimension r. On the other hand, we have $r = \dim(\operatorname{Im} a_1 \otimes \mathbb{R})$ so $r \leq b_1(X) = \dim H^1(X;\mathbb{R})$, and hence this set is nonempty. More precisely, the set $\mathbb{T}_\alpha^G(\mu) \subset \mathbb{T}_\alpha$ is a torus of dimension $b_1(X) - r \geq 0$. □

Lemma 5.2. *The infinitesimal lift $\widetilde{\mathscr{X}}$ defined by (∇, μ) exponentiates to give a linear action of G on L if and only if for any representation $\gamma : S^1 \to G$ we have $M_\gamma = 1$.*

Proof. Given $y \in L$ and $g \in G$, we define $gy \in L$ in the obvious way: let $g = \exp(s)$, where $s \in \mathfrak{g}$, let $\nu_s^y : [0,1] \to L$ be the integral curve of the vector field $\widetilde{\mathscr{X}}^{\nabla,\mu}(s)$ with initial value $\nu_s^y(0) = y$; then $gy := \nu(y; s) = \nu_s^y(1)$.

There are two things to check: that gy is well defined and that the resulting map $G \times L \to L$ is indeed an action of G on L. Observe first that both things are clear when $G = T$ is a torus (see formula (8)). We now sketch how to deal with the general case. Suppose that $s, s' \in \mathfrak{g}$ satisfy $\exp(s) = \exp(s')$. We want to check that, for any $y \in L$, $\nu_s^y(1) = \nu_{s'}^y(1)$. Now, it is easy to prove that there exists some $s'' \in \mathfrak{g}$ such that $\exp(s'') = \exp(s) = \exp(s')$ and such that $[s, s''] = [s', s''] = 0$. Then, applying (8) to some torii T, T' such that $s, s'' \in \operatorname{Lie} T$ and $s', s'' \in \operatorname{Lie} T'$, we deduce that $\nu(y; s) = \nu(y; s'') = \nu(y; s')$. This proves well definedness. Finally, by Baker–Campbell–Haussdorf, ν satisfies $\nu(\nu(y; s); s') = \nu(y; \log(\exp(s)\exp(s')))$ for s, s' small enough, and from this it follows easily that ν defines an action of G on L. □

6. Proofs of the results

We prove the theorems in two steps. First we assume that condition (6) is satisfied. Then we deduce the results in the general case.

6.1. **Proofs of the theorems when (6) holds.**

6.1.1. *Proof of Theorem 1.2.* Combine Lemma 5.2 with the ideas at the end of the proof of Lemma 5.1.

6.1.2. *Proof of Theorem 1.1.* If the action of G lifts to L, then the first equivariant Chern class $c_1^G(L)$ of L is an integral class and provides a lift of $c_1(L)$. Now suppose that $c_1(L) = \iota^*(l)$, where $l \in H^2_G(X;\mathbb{Z})$. Take $\frac{i}{2\pi}(\alpha - \mu) \in \Omega^2_G(X)$ whose cohomology class is equal in $H^2_G(X;\mathbb{R})$ to l. By Theorem 1.2, there is some connection ∇ on L which defines a lift of the action of G to L. Let L_α denote the line bundle L with the action of G. Applying the Chern–Weil construction to equivariant bundles as defined in [BV] for the connection ∇, we deduce that the form $\frac{i}{2\pi}(\alpha - \mu)$ represents $c_1^G(L)$. Now, since we have used de Rham theory, we have lost control of torsion, so that all we know in principle is that

$$c_1^G(L) - l \in \operatorname{Tor} H^2_G(X;\mathbb{Z}).$$

To deduce that $c_1^G(L) = l$, we observe that the restriction of ι^* to $\operatorname{Tor} H^2_G(X;\mathbb{Z})$ is an injection (indeed, $\operatorname{Tor} H^2_G(X;\mathbb{Z}) = \operatorname{Ext}(H_1(X_G;\mathbb{Z}),\mathbb{Z})$, $\operatorname{Tor} H^2(X;\mathbb{Z}) = \operatorname{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z})$ and, since G is connected, $\pi_1(BG) = 0$, so the long exact sequence of homotopy groups for $X \to X_G \to BG$ tells us that $\pi_1(X) \to \pi_1(X_G)$ is exhaustive). So, from $\iota^*(c_1^G(L) - l) = 0$ we deduce that $c_1^G(L) = l$ in $H^2_G(X;\mathbb{Z})$.

To prove that $\iota^{-1}(c_1(L))$ classifies the lifts of the action to L it is enough to check that if G acts on L and $c_1^G(L) = 0$, then L can be G-equivariantly trivialised, i.e., there is an equivariant nowhere vanishing section of L. So assume that G acts on L and $c_1^G(L) = 0$. Take a G-invariant connection ∇ on L, let $\alpha = \nabla^2$ and let μ be the map given by Theorem 3.1. Now, by assumption $[\alpha - \mu] = 0 \in H^2_G(X;i\mathbb{R})$. Since the set of forms representing a fixed cohomology class is connected, we can join $\alpha - \mu$ to $0 \in \Omega^2_G(X;i\mathbb{R})$ through a path $\gamma \subset \Omega^2_G(X;i\mathbb{R})$ all of whose forms represent $0 \in H^2_G(X;i\mathbb{R})$. Fix a trivialisation $L \simeq X \times \mathbb{C}$. It is easy to see, using the proof of Lemma 5.1, that γ can be lifted continuously to give $\forall t$ a connection ∇_t defining a lift to L of the action with Chern–Weil form equal to $\gamma(t)$, in such a way that ∇_1 is the trivial connection on L. So we get a homotopy between the initial action of G on L and the trivial action defined from a trivialisation $L \simeq X \times \mathbb{C}$. Since G is compact, this implies that the initial action of G on L is trivial.

6.2. **Proof of the theorems in the general case.** Suppose that $T = \operatorname{Im} a_1 \cap \operatorname{Tor} H_1(X;\mathbb{Z})$ is nonzero. Let $T' \subset H_1(X;\mathbb{Z})$ be a complementary submodule of T, and let G_T be the connected Lie group which fits in the exact sequence

$$1 \to T \to G_T \xrightarrow{q} G \to 1$$

with $q_* H_1(G_T;\mathbb{Z}) = T'$. The action of G induces an action of G_T on X, and we have a commutative diagram

$$\begin{tikzcd} H^2_G(X;\mathbb{Z}) \arrow[rr, "q"] \arrow[dr, "\iota^*"'] & & H^2_{G_T}(X;\mathbb{Z}) \arrow[dl, "\iota_T^*"] \\ & H^2(X;\mathbb{Z}). & \end{tikzcd}$$

Now, the action of G_T clearly satisfies condition (6), so we may apply the results obtained in the preceeding subsection and get lifts of the action of G_T to L, together with invariant connections.

To prove Theorems 1.1 and 1.2 for the action of G it is enough to check that, if L_α is a G_T bundle isomorphic to L (as bundles over X) such that

$$c_1^{G_T}(L_\alpha) \in q^* H^2_G(X;\mathbb{Z})$$

then the action of G_T on L_α descends to an action of G, or, equivalently, the action of $T \subset G_T$ on L_α is trivial (note that, on the other hand, q^* is injective). This follows from the sequence of maps

$$H^2_G(X;\mathbb{Z}) \xrightarrow{q^*} H^2_{G_T}(X;\mathbb{Z}) \xrightarrow{r^*} H^2(BT;\mathbb{Z}),$$

which is induced by the fibration $BT \to X_{G_T} \to X_G$, and consequently satisfies $r^*q^* = 0$. The map r^* is obtained from the T-equivariant inclusion $x_0 \to X$ (where x_0 is any point). And, since a representation $\rho : T \to \mathbb{C}^*$ is trivial if and only if $c_1^T(ET \times_\rho \mathbb{C}) = 0$, we deduce that if $c_1^{G_T}(L_\alpha) \in q^* H^2_G(X;\mathbb{Z})$ then $r^* c_1^{G_T}(L_\alpha) = 0$ and hence T acts trivially on L_α.

6.3. **Proof of Corollary 1.3.** A theorem of Kirwan (see Proposition 5.8 in [Ki]) says that if G acts in a Hamiltonian fashion on X then there is an isomorphism $H^*_G(X;\mathbb{Q}) \simeq H^*(X;\mathbb{Q}) \otimes H^*(BG;\mathbb{Q})$. In particular, this means that there exists an integer $d \geq 1$ such that if $a \in H^2(X;\mathbb{Z})$ then $da \in \iota^* H^2_G(X;\mathbb{Z})$. So, given the line bundle L, there exists $l \in H^2_G(X;\mathbb{Z})$ such that $c_1(L^d) = \iota^* l$. Let now $\frac{i}{2\pi}(\alpha' - \mu') \in \Omega^2_G(X)$ represent l in $H^2_G(X;\mathbb{R})$, and let ∇_d be a connection on L^d whose curvature is α'. Let $\eta := (\nabla^{\otimes d} - \nabla_d)^G$, where ^{G} means the projection to the invariant subspace $\Omega^1(X; i\mathbb{R})^G$ using the standard averaging trick: $\zeta^G = \frac{1}{|G|}\int_{g\in G} g\zeta$. Then

$$\alpha - \mu := (\alpha' - \mu') + d_{\mathfrak{g}}\eta \in \Omega^2_G(X; i\mathbb{R})$$

represents $-2\pi i l$, and the curvature of ∇ is α. On the other hand, $\operatorname{Im} a_1 = 0$, since any Hamiltonian action of S^1 on a compact manifold has fixed points, and hence the orbits are contractible. Consequently, by Theorem 1.2, the lift $\widetilde{\mathscr{X}}^{\nabla^{\otimes d},\mu}$ exponentiates to an action of G on L which leaves ∇ fixed.

6.4. **Proof of Corollary 1.4.** Let $\mu : X \to \mathfrak{g}^*$ be a moment map for the action of G on X. By Corollary 1.3, it suffices to take any closed $\omega' - \mu' \in \Omega^2_G(X)$ near $\omega - \mu$ and representing a class $[\omega' - \mu'] \in H^2_G(X; 2\pi\mathbb{Q})$.

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